Can the post-Newtonian gravitational waveform of an inspiraling binary be improved by solving the energy balance equation numerically?

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The detection of gravitational waves from inspiraling compact binaries using matched filtering depends crucially on the availability of accurate template waveforms. We determine whether the accuracy of the templates' phasing can be improved by solving the post-Newtonian energy balance equation numerically, rather than (as is normally done) analytically within the post-Newtonian perturbative expansion. By specializing to the limit of a small mass ratio, we find evidence that there is no gain in accuracy.

I. INTRODUCTION AND SUMMARY

Several kilometer-scale interferometric gravitational wave detectors are currently being built, among them the two American LIGO detectors, the French-Italian VIRGO detector, the German-British GEO 600 detector and the Japanese TAMA detector. Gravitational waves from inspiraling compact binaries are among the most promising candidates to be detected. In order to extract a gravitational wave signal from the noisy background the technique of matched filtering [1,2] will be used. One of the drawbacks of matched filtering is that the theoretical templates used must be close to the actual gravitational wave signal in order to detect the signal and estimate its parameters. In the case of a nearly circular inspiral of two point masses the expected gravitational wave signal has the form of a chirp, i.e., a roughly sinusoidal signal with gradually increasing amplitude and frequency. If such a signal is to be detected by matched filtering, high accuracy is needed in the templates; in particular, the phase of the template must closely match the phase of the actual signal. Inspiral templates have been calculated to date up to post-2.5-Newtonian order [3].

Several authors have investigated the question of to what post-Newtonian order does one need to push template computations in order to have an acceptably small template-inaccuracy reduction in event detection rate [4–7]. The result is that post-2-Newtonian templates may be sufficiently accurate to detect neutron star/neutron star binaries [8]; the loss in event rate in this case is $\sim 12\%$ for initial LIGO and $\sim 20\%$ for advanced LIGO [6].

However, there are several motivations for trying to obtain more accurate templates. First, there may well be a high event rate of neutron star/black hole or black hole/black hole inspirals. For these more massive systems the accuracy requirements are more stringent, since the frequency band ($\sim 50-200\,\mathrm{Hz}$) where most of the signal-to-noise is accumulated is in a more relativistic regime for more massive systems. For example, for initial LIGO detectors and for a binary system of a $4M_\odot$ black hole and a $30M_\odot$ black hole, using post-2-Newtonian search templates would allow us to detect only $\sim 35\%$ of the signals that otherwise would be detectable [6]. A second motivation is that one will need high accuracy templates in order to avoid appreciable systematic errors in measurements of the binary's parameters [11–13].

A variety of methods of increasing template accuracy have been pursued in recent years. First, one can compute the templates up to ever-higher post-Newtonian orders; this is arduous but going beyond the current post-2.5-Newtonian templates is feasible. Progress is being made on computing post-3-Newtonian templates; see Ref. [14] and references therein. Second, a celebrated result in this field was the discovery by Damour, Iyer, and Sathyaprakash [11] that using Padé approximants can significantly increase the accuracy of template phasing. Third, Damour and Buonanno [15] have suggested a particular ansatz for obtaining templates containing additional terms of all post-Newtonian orders, starting, say, from post-2-Newtonian templates.

The purpose of this paper is to investigate yet another possible method of improving the accuracy of post-Newtonian templates. The basic idea is very simple. In computing post-2-Newtonian templates, for example, one should strictly speaking discard all terms of post-2.5-Newtonian order (and higher) everywhere in the calculations. To do otherwise would be inconsistent with the perturbation expansion method. Yet, there could be pieces of the calculation for which *retaining* post-2.5-Newtonian (and higher) order terms would lead to improved accuracy. For example, the dominant, $m=2$ piece of the waveform is usually written as

$$h(t) = A(t)\cos\left[\phi^{GW}(t)\right], \tag{1.1}$$

where both the amplitude $A(t)$ and the phase $\phi^{GW}(t)$ have separate post-Newtonian expansions, $A = \sum_j \varepsilon^j A^{(j)}$ and $\phi^{GW} = \sum_j \varepsilon^j \phi^{(j)}$, with ε a formal expansion parameter. Now a perturbation theory purist would insist on inserting the expansion for ϕ^{GW} into Eq. (1.1) and on expanding the cosine using a Taylor expansion.

However, it is well known that the resulting expression is a much poorer representation of the true signal than the original un-expanded form (1.1).

The question then arises: are there other stages in the construction of post-Newtonian templates where one discards higher order terms, which, if retained, might lead to increased accuracy? A natural possibility is the stage in which one goes from the post-Newtonian formulae for the energy flux $F(f) = -dE/dt(f)$ and orbital energy $E(f)$ as functions of gravitational wave frequency f, to the formula for the phase $\phi^{GW}(t)$ of the gravitational waveform. Given analytical formulae for $F(f)$ and $E(f)$ up to some post-Newtonian order, one can either (I) solve analytically for $\phi^{GW}(t)$ within the post-Newtonian approximation, discarding all higher order terms, or (II) one can *numerically solve* the energy balance equation to obtain $\phi^{GW}(t)$. This second procedure effectively generates and retains terms at all post-Newtonian orders, so is strictly speaking inconsistent, but one might hope that it would lead to increased accuracy. We note that the papers [4,5,11] investigating the accuracy of post-Newtonian templates have generally used method (II), whereas the popular data analysis package GRASP [16] used in Ref. [17] uses method (I). The GRASP manual [16] speculates that method (II) might be more accurate.

In this paper we present evidence, based on the limiting case of binaries with small mass ratios, that numerically solving the energy balance equation does not in fact increase the accuracy. We arrive at this conclusion after checking the accuracies of methods (I) and (II) in three different ways. We compare expansion coefficients of the Fourier transform of the waveform; we numerically find the relative error in the Fourier transform of the waveform; and we compute overlaps of templates constructed from both methods with the exact waveform. The result, that the numerical solution of the energy balance equation (method (II)) does not increase the accuracy, is disappointing, but constitutes useful information from the point of view of generating template banks for inspiral searches: there is no motivation in terms of increased event rate to solve numerically for the wave's phasing.

II. METHOD OF CALCULATION

In order to explain our calculation, we first summarize how the waveform's phasing can be computed from the energy flux function $F(f)$ and orbital energy function $E(f)$, where f is gravitational wave frequency [11,5]. Let m_1, m_2 be the masses of the two components of the binary and $m = m_1 + m_2$ be the total mass. Let $\phi(t)$ be the orbital phase of the binary, so that $\phi^{GW}(t) = 2\phi(t)$, where ϕ^{GW} is the phase of the dominant, $m = 2$ piece of the waveform. We define the dimensionless variable

$$v = (\pi m f)^{1/3}. \tag{2.1}$$

[Here and throughout we use units with $G = c = 1$.] The orbital phase $\phi(t)$ is derived from the relation

$$\frac{d\phi}{dt} = \pi f, \tag{2.2}$$

and from the energy balance equation

$$\frac{dE(v)}{dt} = -F(v). \tag{2.3}$$

Equations (2.1) – (2.3) yield a parametric solution for $\phi(t)$ given by

$$\phi(v) = \phi_c - \frac{1}{m}\int_{v_i}^{v} d\bar{v}\bar{v}^3\,\frac{E'(\bar{v})}{F(\bar{v})}, \tag{2.4}$$

and

$$t(v) = t_c - \int_{v_i}^{v} d\bar{v}\frac{E'(\bar{v})}{F(\bar{v})}, \tag{2.5}$$

where ϕ_c, t_c and v_i are constants.

In the restricted post-Newtonian approximation, in which we neglect the $m \neq 2$ multipoles, the gravitational waveform has the form $h(t) = A(t)\cos[\phi^{GW}(t)]$, where $A(t)$ is a slowly varying amplitude. The Fourier transform $\tilde{h}(f)$ of this waveform is $\tilde{h}(f) = B(f)\mathrm{e}^{i\psi(f)}$, where $B(f)$ is some frequency dependent prefactor, and where, in the stationary phase approximation, the phase $\psi(f)$ is given by

$$\psi(f) = 2\pi f t(v) - 2\phi(v) - \frac{\pi}{4}. \tag{2.6}$$

Using Eqs. (2.4) and (2.5) gives the frequency domain phase [5]:

$$\begin{aligned}\psi(f) = {}& 2(t_c/m)v^3 - 2\phi_c - \pi/4 \\ & -\frac{2}{m}\int_{v_i}^{v} d\bar{v}\,\left(v^3 - \bar{v}^3\right)\frac{E'(\bar{v})}{F(\bar{v})}.\end{aligned} \tag{2.7}$$

We note that the corrections to the stationary phase approximation are very small, arising only at post-5-Newtonian order [18], so it is sufficient for our purposes to use the expression (2.7).

Equation (2.7) is the starting point for our analysis. We will investigate the accuracy with which various approximations reproduce the Fourier-domain phase $\psi(f)$, which is the version of the phase function that is most relevant for matched filtering. The two possible calculational methods we consider are (I) to insert post-Newtonian expressions for the functions $E(v)$ and $F(v)$ into Eq. (2.7), and discard all the higher order post-Newtonian terms generated, and (II) to insert post-Newtonian expressions for the functions $E(v)$ and $F(v)$ into Eq. (2.7) and solve exactly for the phase $\psi(f)$, retaining all the higher order post-Newtonian terms generated.

To assess the accuracy of each of these two methods, we specialize to the limit $m_1 m_2/m^2 \to 0$ for which the functions $E(v)$ and $F(v)$ are known [5,11,19]. We then check the accuracy of method (I) and (II) in three ways.

A. Checking the accuracy of methods (I) and (II) by comparing expansion coefficients of $\psi(f)$

The first check is entirely analytical. We expand all the phase functions $\psi(f)$ as post-Newtonian power series in v up to some high order (e.g. post-5.5-Newtonian), and compare the accuracy with which methods (I) and (II) reproduce the coefficients in this power series. While this comparison procedure is less accurate than comparing the phases produced by methods (I) and (II) to the exact numerical phase, it does allow us to check whether there is any indication that method (II) is more accurate than method (I).

In more detail, our comparison procedure works as follows. The post-Newtonian expansions for the functions $E(v)$ and $F(v)$ have the general form

$$E(v) = -\frac{1}{2}\eta m v^2 \left[1 + \sum_{i=1} e_{2i} v^{2i} \right], \tag{2.8}$$

$$F(v) = \frac{32}{5}\eta^2 v^{10} \left[1 + \sum_{i=2} f_i v^i + \sum_{i=6} g_i \ln(v) v^i + \ldots \right], \tag{2.9}$$

where $\eta = m_1 m_2/m^2$ is the dimensionless mass ratio. The ellipses in Eq. (2.9) represent possible terms proportional to $(\ln v)^m$ for $m \geq 2$ which could arise at high post-Newtonian orders. The coefficients e_i, f_i and g_i in Eqs. (2.8) and (2.9) are functions of the mass ratio η. For general mass ratios, the coefficients e_i and f_i in are known up to e_4 and f_5 [3], while for $\eta = 0$ all the e_i coefficients are known [5,11] and the f_i and g_i coefficients are known up to f_{11} and g_{11} [19]. The known coefficients are tabulated in Appendix A.

If we now insert the expansions (2.8) and (2.9) into the formula (2.7) for the phase $\psi(f)$ we obtain

$$\psi(f) = \frac{3v^{-5}}{128\eta} \left[P(v) + \frac{256\eta}{3m} v^8 t_K + \frac{128\eta}{3} v^5 K \right]. \tag{2.10}$$

Here t_K and K are constants which correspond to the initial time and initial phase, and the function $P(v)$ has the expansion

$$P(v) = 1 + \sum_{j=2} \left\{ p_j + q_j \ln(v) + r_j [\ln(v)]^2 + \ldots \right\} v^j. \tag{2.11}$$

The coefficients p_j, q_j and r_j in Eq. (2.11) are functions of the coefficients e_i, f_i and g_i in Eqs. (2.8) and (2.9) for $i \leq j$:

$$p_j = p_j(e_1, \ldots e_j, f_1, \ldots f_j, g_1, \ldots g_j), \tag{2.12}$$

$$q_j = q_j(e_1, \ldots e_j, f_1, \ldots f_j, g_1, \ldots g_j), \tag{2.13}$$

$$r_j = r_j(e_1, \ldots e_j, f_1, \ldots f_j, g_1, \ldots g_j). \tag{2.14}$$

For example, the expressions for the first few p_j's are

$$p_2 = \frac{20(2e_2 - f_2)}{9}, \tag{2.15}$$

$$p_3 = -4 f_3, \tag{2.16}$$

and

$$p_4 = 10(f_2^2 - 2e_2 f_2 + 3e_4 - f_4). \tag{2.17}$$

Suppose now that the functions $E(f)$ and $F(f)$ are known to post-N-Newtonian order. Then the coefficients e_i, f_i and g_i are known for $0 \leq i \leq 2N$. If we now follow the usual method (I) to generate the phase function $\psi(f)$, we obtain an expansion of the form (2.11) where the coefficients are given by

$$\,^{(I)}_{N}p_j = \begin{cases} p_j(e_1, \ldots e_j, f_1, \ldots f_j, g_1, \ldots g_j) & j \leq 2N, \\ 0 & j > 2N, \end{cases} \tag{2.18}$$

$$\,^{(I)}_{N}q_j = \begin{cases} q_j(e_1, \ldots e_j, f_1, \ldots f_j, g_1, \ldots g_j) & j \leq 2N, \\ 0 & j > 2N, \end{cases} \tag{2.19}$$

and

$$\,^{(I)}_{N}r_j = \begin{cases} r_j(e_1, \ldots e_j, f_1, \ldots f_j, g_1, \ldots g_j) & j \leq 2N, \\ 0 & j > 2N. \end{cases} \tag{2.20}$$

Here the superscript (I) means method (I) and the subscript N refers to the post-N-Newtonian approximation. On the other hand, if we use instead the method (II) to generate $\psi(f)$, we obtain an expansion with expansion coefficients

$$\,^{(II)}_{N}p_j = \begin{cases} p_j(e_1, \ldots e_j, f_1, \ldots f_j, g_1, \ldots g_j) & j \leq 2N, \\ p_j(e_1, \ldots e_{2N}, 0, \ldots 0, f_1, \ldots f_{2N}, & \\ \qquad 0, \ldots 0, g_1, \ldots g_{2N}, 0, \ldots 0) & j > 2N, \end{cases} \tag{2.21}$$

$$\,^{(II)}_{N}q_j = \begin{cases} q_j(e_1, \ldots e_j, f_1, \ldots f_j, g_1, \ldots g_j) & j \leq 2N, \\ q_j(e_1, \ldots e_{2N}, 0, \ldots 0, f_1, \ldots f_{2N}, & \\ \qquad 0, \ldots 0, g_1, \ldots g_{2N}, 0, \ldots 0) & j > 2N, \end{cases} \tag{2.22}$$

and

$$\,^{(II)}_{N}r_j = \begin{cases} r_j(e_1, \ldots e_j, f_1, \ldots f_j, g_1, \ldots g_j) & j \leq 2N, \\ r_j(e_1, \ldots e_{2N}, 0, \ldots 0, f_1, \ldots f_{2N}, & \\ \qquad 0, \ldots 0, g_1, \ldots g_{2N}, 0, \ldots 0) & j > 2N. \end{cases} \tag{2.23}$$

Thus, the two methods agree on p_j, q_j and r_j for $j \leq 2N$, but for $j > 2N$ method (I) gives expansion coefficients of zero while method (II) yields coefficients of the form $p_j(e_1, \ldots e_{2N}, 0, \ldots 0, f_1, \ldots f_{2N}, 0, \ldots 0, g_1, \ldots g_{2N}, 0, \ldots 0)$ which differ somewhat from the true values $p_j(e_1, \ldots e_j, f_1, \ldots f_j, g_1, \ldots g_j)$ because of having the coefficients e_i, f_i and g_i set to zero for $2N+1 \leq i \leq j$. We define

$$p_{j,k} = {}^{(II)}_{k/2}p_j, \tag{2.24}$$

and similarly for q_j and r_j, so that $p_{k,k} = p_k$.

As an example, suppose that the functions $E(f)$ and $F(f)$ were known only up to post-1.5-Newtonian order, so that only the coefficients e_2, f_2 and f_3 were known, but not e_4 and f_4. Up second post-Newtonian order the expansion (2.11) has the form

$$P(v) = 1 + p_2 v^2 + p_3 v^3 + p_4 v^4 + O(v^5), \tag{2.25}$$

where the coefficients p_2, p_3 and p_4 are given in Eqs. (2.15) – (2.17) above. How accurately could we determine the coefficients p_2, p_3 and p_4 in this case? Obviously we could compute p_2 and p_3 exactly since they do not depend on e_4 and f_4. However, the coefficient p_4 does depend on e_4 and f_4, and can be written as [cf. Eq. (2.17) above]

$$p_4 = p_{4,3} + \Delta p_{4,3}. \tag{2.26}$$

Here

$$p_{4,3} = {}^{(II)}_{1.5}p_4 = 10\,(f_2^2 - 2e_2 f_2), \tag{2.27}$$

is the piece of p_4 that can be computed from the post-1.5-Newtonian pieces of $E(f)$ and $F(f)$; it is thus nonlinear in the coefficients e_2 and f_2. The value (2.27) is the prediction of method (II), while the method (I) gives instead the value ${}^{(I)}_{1.5}p_4 = 0$. The error term in Eq. (2.26) is given by

$$\Delta p_{4,3} = 10\,(3e_4 - f_4) \tag{2.28}$$

and is linear in the post-2-Newtonian coefficients e_4 and f_4. Using the values of e_2, f_2, e_4 and f_4 listed in Appendix A we find that $\Delta p_{4,3}/p_4 \approx -1.73$ for $\eta = 0$, which is rather large. Hence in this particular example we do not improve the accuracy in the coefficient p_4 by using method (II) rather than method (I).

In general, the question we want to address is whether the approximate coefficient ${}^{(II)}_{N}p_j = p_{j,2N}$ is typically significantly closer to the true coefficient p_j than zero is to p_j, for $j > 2N$, i.e., whether

$$\frac{|p_{j,2N} - p_j|}{p_j} \lesssim \text{(a few tens of percent)} \tag{2.29}$$

(and similarly for q_j and r_j). In Tables I, II and III below we list the values of the true coefficients p_j, q_j and r_j and also the approximate coefficients $p_{j,k}$, $q_{j,k}$ and $r_{j,k}$ for various values of k, computed from the values given in Appendix A using Eqs. (2.8), (2.9) and (2.7). We list the analytic expressions for these approximate coefficients in Appendix B. Examination of Tables I, II and III shows that there is no tendency for the inequality (2.29) to be satisfied.

Therefore method (II) does not seem to lead to a gain in accuracy when compared to method (I) in the test mass case ($\eta \to 0$).

TABLE I. The coefficients $p_{j,k} = {}^{(II)}_{k/2}p_j$ as calculated according to method (II). These coefficients are what one obtains if the orbital energy $E(f)$ and gravitational wave luminosity $F(f)$ as functions of frequency f are known only up to post-$k/2$-Newtonian order. Note that the values of $p_{j,k}$ differ significantly from their true values $p_j = p_{j,j}$ for $k < j$.

method (II)	k=2	k=3	k=4	k=5	k=6	k=7	k=8	k=9	k=10	k=11	true values	
$p_{3,k} \times 10^{-3}$	0	-0.0503									$p_3 \times 10^{-3}$	-0.0503
$p_{4,k} \times 10^{-3}$	0.0821	0.0821	0.0301								$p_4 \times 10^{-3}$	0.0301
$p_{5,k} \times 10^{-3}$	0	0.331	0.331	0.161							$p_5 \times 10^{-3}$	0.161
$p_{6,k} \times 10^{-3}$	-0.609	-3.77	-3.60	-3.60	-0.441						$p_6 \times 10^{-3}$	-0.441
$p_{7,k} \times 10^{-3}$	0	7.59	7.52	2.98	2.98	0.954					$p_7 \times 10^{-3}$	0.954
$p_{8,k} \times 10^{-3}$	-0.502	-7.26	-7.19	-2.91	0.828	0.828	0.995				$p_8 \times 10^{-3}$	0.995
$p_{9,k} \times 10^{-3}$	0	-37.8	-40.2	-28.8	5.61	11.6	11.6	4.43			$p_9 \times 10^{-3}$	4.43
$p_{10,k} \times 10^{-3}$	1.68	43.3	46.5	15.2	-1.76	-12.0	-11.5	-11.5	-8.77		$p_{10} \times 10^{-3}$	-8.77
$p_{11,k} \times 10^{-3}$	0	-76.8	-82.5	-28.4	19.4	26.1	23.9	14.4	14.4	12.3	$p_{11} \times 10^{-3}$	12.3

TABLE II. The coefficients $q_{j,k}$; see caption of Table I.

method (II)	k=2	k=3	k=4	k=5	k=6	k=7	k=8	k=9	k=10	k=11	true values	
$q_{3,k} \times 10^{-3}$	0	0									$q_3 \times 10^{-3}$	0
$q_{4,k} \times 10^{-3}$	0	0	0								$q_4 \times 10^{-3}$	0
$q_{5,k} \times 10^{-3}$	0	0.992	0.992	0.482							$q_5 \times 10^{-3}$	0.482
$q_{6,k} \times 10^{-3}$	0	0	0	0	-0.326						$q_6 \times 10^{-3}$	-0.326
$q_{7,k} \times 10^{-3}$	0	0	0	0	0	0					$q_7 \times 10^{-3}$	0
$q_{8,k} \times 10^{-3}$	1.51	21.8	21.6	8.74	-2.91	-2.91	-3.18				$q_8 \times 10^{-3}$	-3.18
$q_{9,k} \times 10^{-3}$	0	0	0	0	-4.10	-4.10	-4.10	-2.05			$q_9 \times 10^{-3}$	-2.05
$q_{10,k} \times 10^{-3}$	0	0	0	0	1.95	1.95	0.702	0.702	0.235		$q_{10} \times 10^{-3}$	0.235
$q_{11,k} \times 10^{-3}$	0	0	0	0	-6.00	-6.00	-3.05	-0.356	-0.356	-1.41	$q_{11} \times 10^{-3}$	-1.41

TABLE III. The coefficients $r_{j,k}$; see caption of Table I. These coefficients vanish for $j \leq 7$.

method (II)	k=2	k=3	k=4	k=5	k=6	k=7	k=8	k=9	k=10	k=11	true values	
$r_{8,k} \times 10^{-3}$	0	0	0	0	1.29	1.29	0.584				$r_8 \times 10^{-3}$	0.584
$r_{9,k} \times 10^{-3}$	0	0	0	0	0	0	0	0			$r_9 \times 10^{-3}$	0
$r_{10,k} \times 10^{-3}$	0	0	0	0	0	0	0	0	0		$r_{10} \times 10^{-3}$	0
$r_{11,k} \times 10^{-3}$	0	0	0	0	0	0	0	0	0	0	$r_{11} \times 10^{-3}$	0

B. Checking the accuracy of methods (I) and (II) numerically

Next we perform a direct numerical check by comparing the phases produced by methods (I) and (II) to the exact numerical phase. Note that the phase $\psi(f)$ in Eq. (2.7) is not directly observable, since it contains the unknown integration constants ϕ_c and t_c, i.e. $\psi(f)$ is determined only up to a linear function in f. Hence the relevant quantity to consider is

$$\psi''(f) = \frac{d^2}{df^2}\psi(f) = -\frac{2\pi^2 m}{3v^2}\frac{E'(v)}{F(v)}. \qquad (2.30)$$

In the test mass case $E'(v)$ and $F(v)$ are known exactly [5,11,19], and we can therefore find the exact $\psi''(f)$.

Now suppose $E'(v)$ and $F(v)$ are only known up to post-$k/2$-Newtonian order. If method (I) is used $\psi''(f)$ is found to be

$$ {}^{(I)}_{k/2}\psi''(f) = -\frac{2\pi^2 m}{3v^2}\left[\frac{E'(v)}{F(v)}\right]_k. \qquad (2.31)$$

On the other hand method (II) yields

$$ {}^{(II)}_{k/2}\psi''(f) = -\frac{2\pi^2 m}{3v^2}\frac{[E'(v)]_k}{[F(v)]_k}. \qquad (2.32)$$

Here $[...]_k$ denotes the powerseries of the expression inside the brackets with terms kept up to order v^k.

It is convenient to define the logarithmic relative errors

$$\ln\left|\frac{{}^{(I)}_{k/2}\psi''(f) - \psi''(f)}{\psi''(f)}\right| = \ln\left|\frac{\left[\frac{E'(v)}{F(v)}\right]_k - \frac{E'(v)}{F(v)}}{\frac{E'(v)}{F(v)}}\right| \qquad (2.33)$$

and

$$\ln\left|\frac{{}^{(II)}_{k/2}\psi''(f) - \psi''(f)}{\psi''(f)}\right| = \ln\left|\frac{\frac{[E'(v)]_k}{[F(v)]_k} - \frac{E'(v)}{F(v)}}{\frac{E'(v)}{F(v)}}\right|. \qquad (2.34)$$

These errors are shown in Figures 1 - 5. It can be seen that there is no systematic tendency for method (II) to perform better than method (I). At post-2.5 and post-4-Newtonian order method (I) does better than method (II) for all v, while at post-3 and post-3.5-Newtonian order method (II) is more accurate than method (I). We would expect the same trend to hold for general values of the mass ratio η.

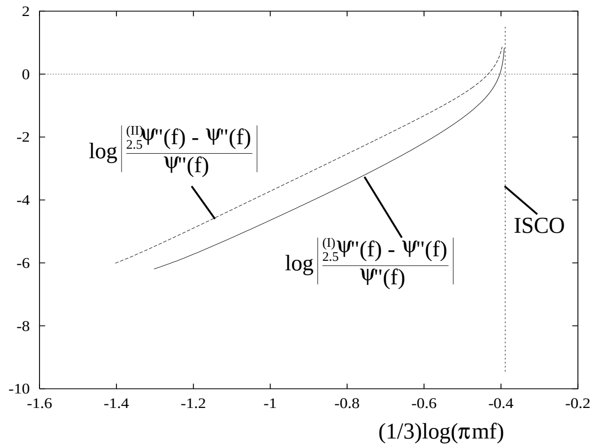

FIG. 1. The errors in the phase $\psi(f)$ of the Fourier transformed waveform produced by methods (I) and (II) in the test mass limit. Plotted here are the logarithms of the relative errors in the second derivative $\psi''(f)$, for the case when the energy $E(f)$ and gravitational wave luminosity $F(f)$ are known only up to post-2.5-Newtonian order. The horizontal axis is $\log(\pi m f)/3$ where m is the total mass of the system and f is gravitational wave frequency. The line denoted ISCO indicates the location of the innermost stable circular orbit. It can be seen that method (I) is more accurate for all frequencies f.

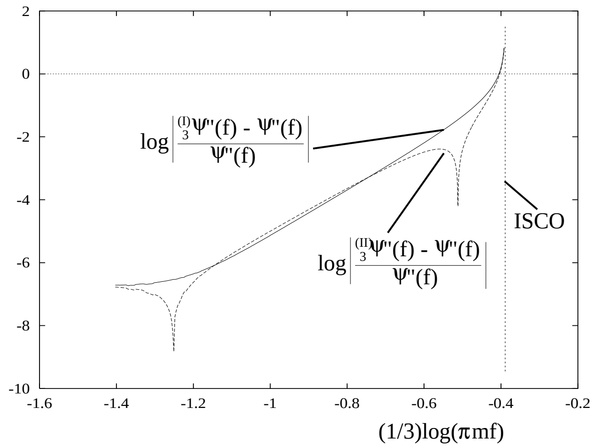

FIG. 2. The errors in the phase $\psi(f)$ when $E(f)$ and $F(f)$ are known only up to post-3-Newtonian order; see caption of Fig. 1. In this case method (II) is overall more accurate.

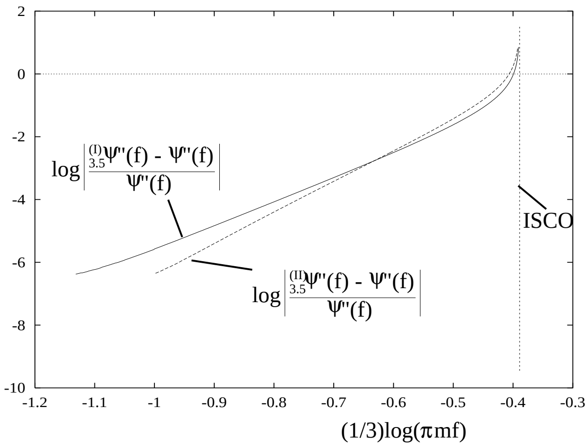

FIG. 3. The errors in the phase $\psi(f)$ when $E(f)$ and $F(f)$ are known only up to post-3.5-Newtonian order; see caption of Fig. 1. In this case method (II) is more accurate for most frequencies f.

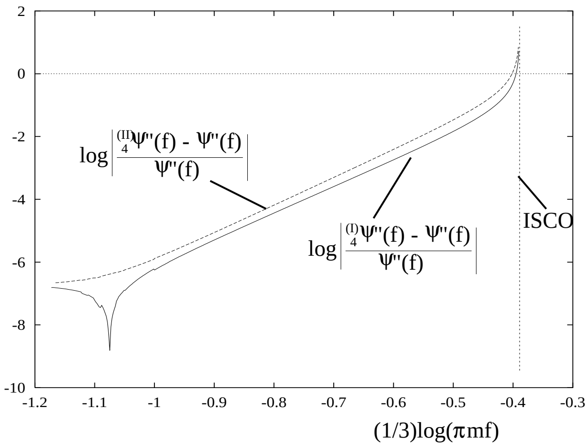

FIG. 4. The errors in the phase $\psi(f)$ when $E(f)$ and $F(f)$ are known only up to post-4-Newtonian order; see caption of Fig. 1. In this case method (I) is more accurate for all frequencies f.

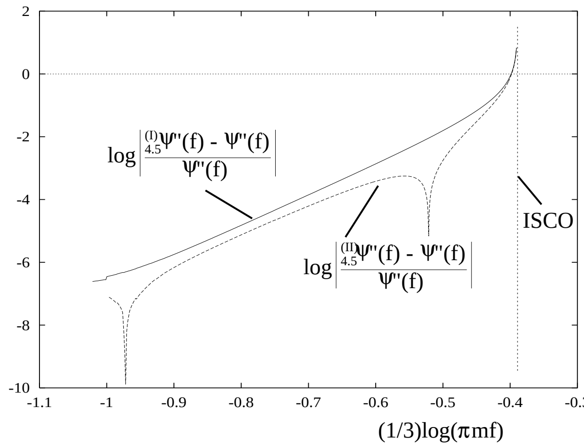

FIG. 5. The errors in the phase $\psi(f)$ when $E(f)$ and $F(f)$ are known only up to post-4.5-Newtonian order; see caption of Fig. 1. In this case method (II) is more accurate for all frequencies f.

C. Overlaps of templates constructed by methods (I) and (II) with the exact signal

So far we have only considered how accurately the phase $\psi(f)$ is generated by methods (I) and (II). In this subsection we use the phases ${}^{(I)}_{k/2}\psi(f)$ and ${}^{(II)}_{k/2}\psi(f)$ to construct gravitational wave templates and then compute the templates' overlap with the exact waveform computed from the exact $\psi(f)$.

We use the restricted post-Newtonian approximation and neglect the $m \neq 2$ multipoles. Thus the Fourier transform $\tilde{h}(f)$ of the exact waveform is given by [4]

$$\tilde{h}(f) \propto f^{-7/6} \mathrm{e}^{i\psi(f)}. \tag{2.35}$$

Similarly we use methods (I) and (II) to construct the templates

$${}^{(I)}_{k/2}\tilde{h}(f) \propto f^{-7/6} \mathrm{e}^{i{}^{(I)}_{k/2}\psi(f)} \tag{2.36}$$

and

$${}^{(II)}_{k/2}\tilde{h}(f) \propto f^{-7/6} \mathrm{e}^{i{}^{(II)}_{k/2}\psi(f)}. \tag{2.37}$$

Next we compute Apostolatos' [20] fitting factor (FF) to determine the templates' accuracy. The fitting factor is the ratio of the signal-to-noise ratio obtained with the imperfect template, to the signal-to-noise ratio that a perfect template would yield. The fitting factor can take values from zero to one, with unity indicating a perfect template. It is obtained from the ambiguity function

$$A = \frac{\left({}^{(I/II)}_{k/2}h, h\right)}{\sqrt{\left({}^{(I/II)}_{k/2}h, {}^{(I/II)}_{k/2}h\right)(h,h)}} \tag{2.38}$$

by maximizing over the template parameters, i.e.

$$FF = \max_{\phi_c, t_c} A. \tag{2.39}$$

Notice that we hold the masses fixed in the maximization procedure: the templates and signal correspond to binaries of the same two masses. Here we have introduced the inner product

$$(s, h) = 2 \int_0^\infty \frac{\tilde{s}(f)^* \tilde{h}(f) + \tilde{s}(f) \tilde{h}(f)^*}{S_n(f)} df, \tag{2.40}$$

where $S_n(f)$ is the spectral density of the detector noise. The noise curve $S_n(f)$ used here is the Cutler-Flanagan fit [21] for the advanced LIGO. The largest contribution to the overlaps comes from the frequency band between 40 Hz and 200 Hz.

We compute the fitting factors for several different choices of m_1 and m_2 in order to get an indication of what might happen for general mass ratios, even though our results apply strictly only to the test mass limit $\eta \to 0$. The resulting fitting factors are listed in tables IV, V and VI. Examination shows close agreement with the error plots of ${}^{(I)}_{k/2}\psi''(f)$ and ${}^{(II)}_{k/2}\psi''(f)$ in Figs. 1 – 5. At post-Newtonian orders where ${}^{(I)}_{k/2}\psi''(f)$ has an error smaller than the error in ${}^{(II)}_{k/2}\psi''(f)$ everywhere (e.g. post-2.5), method (I) always wins, and vice versa (e.g. post-4.5). On the other hand, at post-Newtonian orders where the error lines cross (e.g. post-3.5), the method with the smaller error in the v-region $[v = (\pi m f)^{1/3}]$ selected by the sensitive frequency band of the detector and the total mass m yields a larger fitting factor.

Again we find that there is no systematic tendency for method (II) to be more accurate than method (I). Therefore method (II) does not lead to a gain in accuracy when compared to method (I). Our conclusion applies only to the limit $\eta \to 0$, but we do not anticipate a different result for the general case.

TABLE IV. The fitting factor (FF) at post-$k/2$-Newtonian order for gravitational wave templates constructed by method (I) and (II). A fitting factor of unity indicates a perfect template. This table shows FF in the case of $m_1 = m_2 = 1.4 M_\odot$.

k	FF-method (I)	FF-method (II)
4	0.495960	0.710421
5	0.963230	0.596752
6	0.985540	0.981689
7	0.995678	0.998342
8	0.998731	0.997278
9	0.999264	0.999851
10	0.999661	0.999953
11	0.999911	0.999964

TABLE V. The fitting factor (FF) for method (I) and (II) in the case of $m_1 = 1.4 M_\odot$ and $m_2 = 10 M_\odot$.

k	FF-method (I)	FF-method (II)
4	0.329050	0.871888
5	0.787947	0.341547
6	0.826911	0.872141
7	0.926867	0.930399
8	0.967903	0.919188
9	0.978698	0.995905
10	0.980767	0.990105
11	0.983868	0.992731

TABLE VI. The fitting factor (FF) for method (I) and (II) in the case of $m_1 = m_2 = 10 M_\odot$.

k	FF-method (I)	FF-method (II)
4	0.483132	0.908446
5	0.895806	0.474364
6	0.896781	0.976451
7	0.959834	0.947991
8	0.968440	0.948341
9	0.972900	0.999764
10	0.975185	0.994083
11	0.987979	0.995887

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APPENDIX A: COEFFICIENTS IN EXPANSIONS OF ENERGY AND ENERGY FLUX FUNCTIONS

1. The coefficients e_i and f_i up to post-2.5-Newtonian order

The coefficients in Eqs. (2.8) and (2.9) up to 2.5 post-Newtonian order as given by Blanchet [3] are:

$$e_2 = -\frac{9+\eta}{12}, \tag{A1}$$

$$e_4 = -\frac{27-19\eta+\eta^2/3}{8}, \tag{A2}$$

$$f_2 = -\frac{1247}{336} - \frac{35\eta}{12}, \tag{A3}$$

$$f_3 = 4\pi, \tag{A4}$$

$$f_4 = -\frac{44711}{9072} + \frac{9271\eta}{504} + \frac{65\eta^2}{18}, \tag{A5}$$

and

$$f_5 = -\left(\frac{8191}{672} + \frac{535\eta}{24}\right)\pi. \tag{A6}$$

2. The coefficients e_i, f_i and g_i up to post-5.5-Newtonian order

The remaining coefficients in Eqs. (2.8) and (2.9) have been given up to 5.5 post-Newtonian order in the test mass limit in Ref. [19]. These are

$$e_6 = -\frac{675}{64}, \tag{A7}$$

$$e_8 = -\frac{3969}{128}, \tag{A8}$$

$$e_{10} = -\frac{45927}{512}, \tag{A9}$$

$$f_6 = \frac{6643739519}{69854400} - \frac{1712\gamma}{105} + \frac{16\pi^2}{3} - \frac{3424\log(2)}{105}, \tag{A10}$$

$$f_7 = -\frac{16285\pi}{504}, \tag{A11}$$

$$f_8 = -\frac{323105549467}{3178375200} + \frac{232597\gamma}{4410} - \frac{1369\pi^2}{126} + \frac{39931\log(2)}{294} - \frac{47385\log(3)}{1568}, \tag{A12}$$

$$f_9 = \frac{265978667519\pi}{745113600} - \frac{6848\gamma\pi}{105} - \frac{13696\pi\log(2)}{105}, \tag{A13}$$

$$f_{10} = -\frac{2500861660823683}{2831932303200} + \frac{916628467\gamma}{7858620} - \frac{424223\pi^2}{6804} - \frac{83217611\log(2)}{11226600} + \frac{47385\log(3)}{196}, \tag{A14}$$

$$f_{11} = \frac{8399309750401\pi}{101708006400} + \frac{177293\gamma\pi}{1176} + \frac{8521283\pi\log(2)}{17640} - \frac{142155\pi\log(3)}{784}, \tag{A15}$$

$$g_6 = -\frac{1712}{105}, \tag{A16}$$

$$g_7 = 0, \tag{A17}$$

$$g_8 = \frac{232597}{4410}, \tag{A18}$$

$$g_9 = -\frac{6848\pi}{105}, \tag{A19}$$

$$g_{10} = \frac{916628467}{7858620}, \tag{A20}$$

and

$$g_{11} = \frac{177293\pi}{1176}. \tag{A21}$$

APPENDIX B: ANALYTIC EXPRESSIONS FOR THE COEFFICIENTS IN PHASE EXPANSION IN VARIOUS ORDERS OF APPROXIMATION

1. The coefficients $p_{j,n}$, $q_{j,n}$ and $r_{j,n}$ up to post-2.5-Newtonian order for general mass ratios

Here we list the analytic expressions for the coefficients $p_{j,n}$, $q_{j,n}$ and $r_{j,n}$ up to post-2.5-Newtonian order for $\eta \neq 0$.

$$p_{2,2} = \frac{3715}{756} + \frac{55\,\eta}{9} \tag{B1}$$

$$p_{3,2} = 0 \tag{B2}$$

$$p_{3,3} = -16\,\pi \tag{B3}$$

$$p_{4,2} = p_{4,3} = \frac{5\,\left(926521 + 1880368\,\eta + 905520\,\eta^2\right)}{56448} \tag{B4}$$

$$p_{4,4} = \frac{15293365}{508032} + \frac{27145\,\eta}{504} + \frac{3085\,\eta^2}{72} \tag{B5}$$

$$p_{5,2} = 0 \tag{B6}$$

$$p_{5,3} = p_{5,4} = \frac{20\,\left(995 + 952\,\eta\right)\,\pi}{189} \tag{B7}$$

$$p_{5,5} = \frac{5\,\left(7729 + 252\,\eta\right)\,\pi}{756} \tag{B8}$$

$$q_{2,n} = q_{3,n} = q_{4,n} = 0 \tag{B9}$$

$$q_{5,2} = 0 \tag{B10}$$

$$q_{5,3} = q_{5,4} = \frac{20\,\left(995 + 952\,\eta\right)\,\pi}{63} \tag{B11}$$

$$q_{5,5} = \frac{38645\,\pi}{252} + 5\,\eta\,\pi \tag{B12}$$

$$r_{2,n} = r_{3,n} = r_{4,n} = r_{5,n} = 0 \tag{B13}$$

2. The coefficients $p_{j,n}$, $q_{j,n}$ and $r_{j,n}$ up to post-5.5-Newtonian order in the test mass limit

Here we list analytic expressions for the remaining coefficients $p_{j,n}$, $q_{j,n}$ and $r_{j,n}$ up to post-5.5-Newtonian order in the test mass limit.

$$p_{6,2} = -\frac{5776858435}{9483264} \tag{B14}$$

$$p_{6,3} = -\frac{5776858435}{9483264} - 320\,\pi^2 \tag{B15}$$

$$p_{6,4} = p_{6,5} = -\frac{37674179035}{85349376} - 320\,\pi^2 \tag{B16}$$

$$p_{6,6} = \frac{10817850546611}{4694215680} - \frac{6848\,\gamma}{21} - \frac{640\,\pi^2}{3} - \frac{13696\,\log(2)}{21} \tag{B17}$$

$$p_{7,2} = 0 \tag{B18}$$

$$p_{7,3} = \frac{5680085\,\pi}{2352} \tag{B19}$$

$$p_{7,4} = \frac{152000375\,\pi}{63504} \tag{B20}$$

$$p_{7,5} = p_{7,6} = \frac{241249475\,\pi}{254016} \tag{B21}$$

$$p_{7,7} = \frac{77096675\,\pi}{254016} \tag{B22}$$

$$p_{8,2} = -\frac{7203742468445}{14338695168} \tag{B23}$$

$$p_{8,3} = -\frac{7203742468445}{14338695168} - \frac{43160\,\pi^2}{63} \tag{B24}$$

$$p_{8,4} = -\frac{499400855271485}{1161434308608} - \frac{43160\,\pi^2}{63} \tag{B25}$$

$$p_{8,5} = -\frac{499400855271485}{1161434308608} - \frac{47570\,\pi^2}{189} \tag{B26}$$

$$p_{8,6} = p_{8,7} = \frac{35381221594107617}{12775777394688} - \frac{1703440\,\gamma}{3969} - \frac{63110\,\pi^2}{567} - \frac{3406880\,\log(2)}{3969} \tag{B27}$$

$$p_{8,8} = \frac{2496799162103891233}{830425530654720} - \frac{36812\,\gamma}{189} - \frac{90490\,\pi^2}{567} - \frac{1011020\,\log(2)}{3969} - \frac{26325\,\log(3)}{196} \tag{B28}$$

$$p_{9,2} = 0 \tag{B29}$$

$$p_{9,3} = \frac{-6756514105\,\pi}{1185408} - 640\,\pi^3 \tag{B30}$$

$$p_{9,4} = \frac{-23087048755\,\pi}{3556224} - 640\,\pi^3 \tag{B31}$$

$$p_{9,5} = \frac{-971321608855\,\pi}{341397504} - 640\,\pi^3 \tag{B32}$$

$$p_{9,6} = \frac{121130241969551\,\pi}{18776862720} - \frac{27392\,\gamma\,\pi}{21} - \frac{640\,\pi^3}{3} - \frac{54784\,\pi\,\log(2)}{21} \tag{B33}$$

$$p_{9,7} = p_{9,8} = \frac{157063289889551\,\pi}{18776862720} - \frac{27392\,\gamma\,\pi}{21} - \frac{640\,\pi^3}{3} - \frac{54784\,\pi\,\log(2)}{21} \tag{B34}$$

$$p_{9,9} = \frac{90036665674763\,\pi}{18776862720} - \frac{13696\,\gamma\,\pi}{21} - \frac{640\,\pi^3}{3} - \frac{27392\,\pi\,\log(2)}{21} \tag{B35}$$

$$p_{10,2} = \frac{1796613371630183}{1070622572544} \tag{B36}$$

$$p_{10,3} = \frac{1796613371630183}{1070622572544} + \frac{1240765\,\pi^2}{294} \tag{B37}$$

$$p_{10,4} = \frac{54094086068862461}{28906809458688} + \frac{11956093\,\pi^2}{2646} \tag{B38}$$

$$p_{10,5} = \frac{54094086068862461}{28906809458688} + \frac{458972531\,\pi^2}{338688} \tag{B39}$$

$$p_{10,6} = -\frac{40278025476453416665}{317974904045568} + \frac{650561605\,\gamma}{333396} + \frac{312163997\,\pi^2}{435456} + \frac{650561605\,\log(2)}{166698} \tag{B40}$$

$$p_{10,7} = -\frac{40278025476453416665}{317974904045568} + \frac{650561605\,\gamma}{333396} - \frac{138083683\,\pi^2}{435456} + \frac{650561605\,\log(2)}{166698} \tag{B41}$$

$$p_{10,8} = p_{10,9} = -\frac{236001272110671078 43}{1878942614814720} + \frac{116990189\,\gamma}{166698} - \frac{181984501\,\pi^2}{3048192} + \frac{228376895\,\log(2)}{333396} + \frac{15716025\,\log(3)}{21952} \tag{B42}$$

$$p_{10,10} = -\frac{141220699543295798 2751}{126306697995878400} + \frac{6470582647\,\gamma}{27505170} + \frac{578223115\,\pi^2}{3048192} + \frac{53992839431\,\log(2)}{55010340} - \frac{5512455\,\log(3)}{21952} \tag{B43}$$

$$p_{11,2} = 0 \tag{B44}$$

$$p_{11,3} = \frac{-40905234824185\,\pi}{7169347584} - \frac{358720\,\pi^3}{189} \tag{B45}$$

$$p_{11,4} = \frac{-1456611391753955\,\pi}{193572384768} - \frac{358720\,\pi^3}{189} \tag{B46}$$

$$p_{11,5} = \frac{-1211268636338065\,\pi}{387144769536} - \frac{112990\,\pi^3}{189} \tag{B47}$$

$$p_{11,6} = \frac{41090763354419749\,\pi}{4258592464896} - \frac{7577740\,\gamma\,\pi}{3969} + \frac{15130\,\pi^3}{567} - \frac{15155480\,\pi\,\log(2)}{3969} \tag{B48}$$

$$p_{11,7} = \frac{50239568645429749\,\pi}{4258592464896} - \frac{7577740\,\gamma\,\pi}{3969} + \frac{15130\,\pi^3}{567} - \frac{15155480\,\pi\,\log(2)}{3969} \tag{B49}$$

$$p_{11,8} = \frac{3146788245124283189\,\pi}{276808510218240} - \frac{183628\,\gamma\,\pi}{189} - \frac{94390\,\pi^3}{567} - \frac{5572040\,\pi\,\log(2)}{3969} - \frac{26325\,\pi\,\log(3)}{49} \tag{B50}$$

$$p_{11,9} = p_{11,10} = \frac{1846304168796859019\,\pi}{276808510218240} - \frac{449308\,\gamma\,\pi}{3969} - \frac{94390\,\pi^3}{567} + \frac{1241720\,\pi\,\log(2)}{3969} - \frac{26325\,\pi\,\log(3)}{49} \tag{B51}$$

$$p_{11,11} = \frac{1795505143426433771\,\pi}{276808510218240} - \frac{3558011\,\gamma\,\pi}{7938} - \frac{94390\,\pi^3}{567} - \frac{862549\,\pi\,\log(2)}{1134} - \frac{26325\,\pi\,\log(3)}{196} \tag{B52}$$

$$q_{6,2} = q_{6,3} = q_{6,4} = q_{6,5} = 0 \tag{B53}$$

$$q_{6,6} = -\frac{6848}{21} \tag{B54}$$

$$q_{7,n} = 0 \tag{B55}$$

$$q_{8,2} = \frac{7203742468445}{4779565056} \tag{B56}$$

$$q_{8,3} = \frac{7203742468445}{4779565056} + \frac{43160\,\pi^2}{21} \tag{B57}$$

$$q_{8,4} = \frac{499400855271485}{387144769536} + \frac{43160\,\pi^2}{21} \tag{B58}$$

$$q_{8,5} = \frac{499400855271485}{387144769536} + \frac{47570\,\pi^2}{63} \tag{B59}$$

$$q_{8,6} = q_{8,7} = -\frac{37208950681636577}{4258592464896} + \frac{1703440\,\gamma}{1323} + \frac{63110\,\pi^2}{189} + \frac{3406880\,\log(2)}{1323} \tag{B60}$$

$$q_{8,8} = -\frac{2550713843998885153}{276808510218240} + \frac{36812\,\gamma}{63} + \frac{90490\,\pi^2}{189} + \frac{1011020\,\log(2)}{1323} + \frac{78975\,\log(3)}{196} \tag{B61}$$

$$q_{9,2} = q_{9,3} = q_{9,4} = q_{9,5} = 0 \tag{B62}$$

$$q_{9,6} = q_{9,7} = q_{9,8} = \frac{-27392\,\pi}{21} \tag{B63}$$

$$q_{9,9} = \frac{-13696\,\pi}{21} \tag{B64}$$

$$q_{10,2} = q_{10,3} = q_{10,4} = q_{10,5} = 0 \tag{B65}$$

$$q_{10,6} = q_{10,7} = \frac{650561605}{333396} \tag{B66}$$

$$q_{10,8} = q_{10,9} = \frac{116990189}{166698} \tag{B67}$$

$$q_{10,10} = \frac{6470582647}{27505170} \tag{B68}$$

$$q_{11,2} = q_{11,3} = q_{11,4} = q_{11,5} = 0 \tag{B69}$$

$$q_{11,6} = q_{11,7} = \frac{-7577740\,\pi}{3969} \tag{B70}$$

$$q_{11,8} = \frac{-183628\,\pi}{189} \tag{B71}$$

$$q_{11,9} = q_{11,10} = \frac{-449308\,\pi}{3969} \tag{B72}$$

$$q_{11,11} = \frac{-3558011\,\pi}{7938} \tag{B73}$$

$$r_{6,n} = r_{7,n} = 0 \tag{B74}$$

$$r_{8,2} = r_{8,3} = r_{8,4} = r_{8,5} = 0 \tag{B75}$$

$$r_{8,6} = r_{8,7} = \frac{1703440}{1323} \tag{B76}$$

$$r_{8,8} = \frac{36812}{63} \tag{B77}$$

$$r_{9,n} = r_{10,n} = r_{11,n} = 0 \tag{B78}$$

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